Continuous r.v.'s
Uniform Distribution
Expected Values of Functions of a r.v.
Normal Distribution
Normal Approximation for Binomial
Exponential Dist.
Joint Distributions
Linear Combinations
Examples

Ch. 5 - Continuous Random Variables and Probability Distributions

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Spring, 2016

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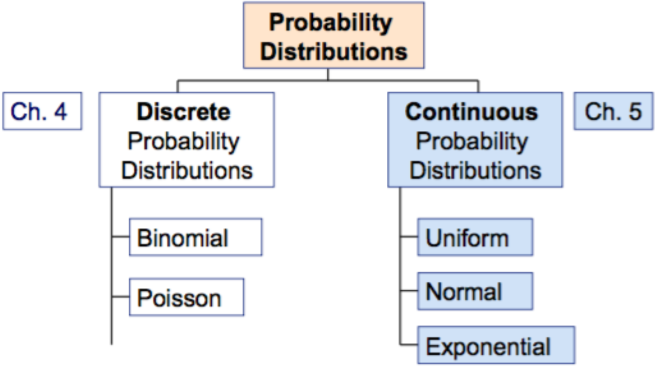
Probability Distributions



Continuous Random Variables

- A continuous random variable is a variable that can assume any value in an interval.
- Examples:
  - thickness of an item
  - time required to complete a task
  - temperature of a solution
  - height, in inches
- These can potentially take on any value, depending only on the ability to measure accurately.

Probability Density Function

- Recall in discrete case, we used the notation $P(X = x^*)$ to denote the probability that the r.v. X assumes a particular value x^*.
- For continuous random variables X, writing $P(X = x^*)$ would be meaningles. This is the most important technical difference between discrete and continuous r.v.'s. Let us see why...
  - For any r.v. if you add up the probabilities for all possible values that the r.v. can assume you should get 1, i.e.

$$\sum_{x^*} P(X = x^*) = 1$$

  - But if X is a continuous r.v. how many possible values are there? **Continuum!**
  - Continuum means even more than infinity, but let us aside this, practically suppose it means infinite and also assume that the least likely value of the r.v. is x^*.
  - Suppose $P(X = x^*) = \varepsilon$, where ε is very very small number.

Probability Density Function

- Suppose $P(X = x^*) = \varepsilon$, where ε is very very small number.
  - But since there are infinite number of possible values that X can assume, each with at least ε probability, we have

$$\sum_x P(X = x) \geq \infty \cdot \varepsilon = \infty$$

  - We got this contradiction because we assumed $P(X = x^*) = \varepsilon$, no matter how small ε is.
- **Bottom Line:** We cannot assign probabilities to the individual that a continuous r.v. can assume!
- Instead we are going to assign **probabilities to every possible interval** that X possibly can fall into.

Probability Density Function

- The probability density function, $f(x)$, of random variable X has the following properties:
  1. $f(x) > 0$ for all values x
  2. The area under the probability density function f(x) over all possible values of X is equal to 1

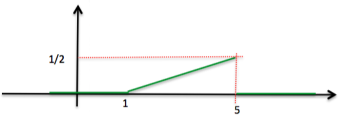
$$P(x_{min} < X < x_{max}) = \int_{x_{min}}^{x_{max}} f(x)\, dx = 1$$

- The probability that X lies between two values is the area under graph of the density function between these two values:

$$P(a < X < b) = \int_{a}^{b} f(x)dx$$

Example 1

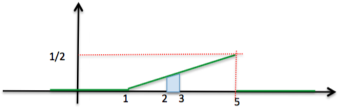
$$f(x) = \begin{cases} \frac{1}{8}(x-1) & 1 \leq x \leq 5 \\ 0 & \text{for all other} \end{cases}$$



- Check whether $f(x)$ is a p.d.f.?

1. $f(x) > 0$ for all values of x. Yes!
2. $P(1 < X < 5) = \int_1^5 \frac{1}{8}(x-1)dx = \frac{4 \cdot \frac{1}{2}}{2} = 1$. Yes!

- Let us also find the probability that the r.v. X will fall into 2 and 3:



$$P(2 < X < 3) = \int_2^3 \frac{1}{8}(x-1)dx = \frac{3}{16}$$

Cumulative Distribution Function

- The cumulative distribution function (c.d.f.) of X, denoted by $F(x)$, expresses the probability that the random variable X does not exceed $X = x$:

$$F(x) = P(X \leq x) = \int_{x_{min}}^{x} f(s)\, ds$$

where $f(s)$ is the p.d.f. of X.

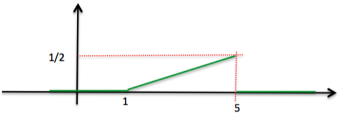
- The following property of $c.d.f.$ makes the calculation of probabilities a lot easier: *Let a and b be two possible values of X, with $a < b$, then the probability that X lies between a and b is*

$$P(a < X < b) = \int_{a}^{b} f(x)\, dx = F(b) - F(a)$$

- Let us see how we could compute the probability in the previous example.

Example 1 cont'd ...

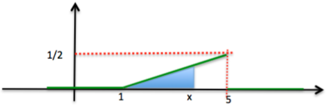
$$f(x) = \begin{cases} \frac{1}{8}(x-1) & 1 \leq x \leq 5 \\ 0 & \text{for all other} \end{cases}$$



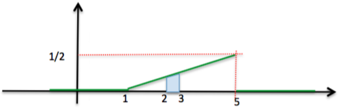
- C.d.f. of X:

$$P(1 < X < x) = \int_1^x \frac{1}{8}(s-1)ds$$

$$= \frac{1}{16}(x-1)^2$$



- Let us also find the probability that the r.v. X will fall into 2 and 3:



$$P(2 < X < 3) = F(3) - F(2)$$

$$= \frac{1}{16}(3-1)^2 - \frac{1}{16}(2-1)^2 = \frac{3}{16}$$

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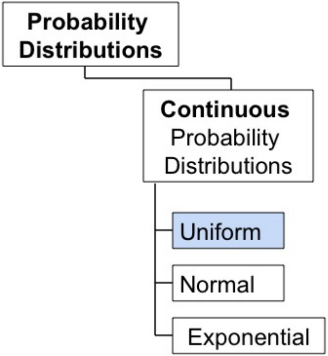
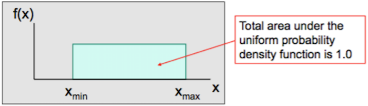
The Uniform Distribution



The Uniform Distribution

- The Uniform Distribution is a probability distribution that has equal probabilities for all equal-width intervals within the range of the random variable:



- The probability distribution function (p.d.f.) for the Uniform Distribution:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where a and b are minimum and maximum values of X, respectively.

Expected Value and Variance of a Continuous RV

Expected Value of a continuous random variable X, with probability density function $f(x)$ is given by

$$E(X) = \int_{x_{min}}^{x_{max}} x \cdot f(x) dx$$

Notation: $\mu_x = E(X)$

Variance of a continuous random variable X, with probability density function $f(x)$ is given by

$$V(X) = \int_{x_{min}}^{x_{max}} \left(x - \mu_x\right)^2 \cdot f(x) dx$$

Notation: $\sigma_x^2 = V(X)$

Standard Deviation of X is defined as the square root of the variance, therefore with this notation it will be simply σ.

Example 1 Cont'd

Let us calculate μ_x and σ_x^2 for the r.v. in Example 1.

$$\begin{aligned}
\mu_x &= \int_{x_{min}}^{x_{max}} x \cdot f(x)dx \\
&= \int_1^5 x \cdot \frac{1}{8}(x-1)dx = \frac{1}{8}\int_1^5 (x^2 - x)dx \\
&= \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2}{2} \,\Big|_1^5 \right] = 3.7
\end{aligned}$$

$$\begin{aligned}
\sigma_x^2 &= \int_{x_{min}}^{x_{max}} \left(x - \mu_x\right)^2 \cdot f(x)dx \\
&= \int_1^5 \left(x - 3.7\right)^2 \frac{1}{8}(x-1)dx \\
&= \frac{1}{8}\int_1^5 \left(x - 3.7\right)^2 (x-1)dx \\
&= \frac{1}{8}\int_1^5 \left(x^3 - 8.4x^2 + 20.8x - 13.4\right)dx = 24/40
\end{aligned}$$

Mean and Variance of the Uniform Distribution

Let a r.v. X be uniformly distributed over the interval [a,b]:

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{for all other x} \end{cases}$$

Then μ_x and σ_x^2 can be computed as

$$\begin{aligned} \mu_x &= \int_a^b x \cdot \frac{1}{b-a} dx \\ &= \frac{1}{b-a} \int_a^b x dx = \frac{1}{b-a} \Big[\frac{x^2}{2} \Big]_a^b \\ &= \frac{a+b}{2} \end{aligned}$$

$$\begin{aligned} \sigma_x^2 &= \int_a^b (x-\mu_x)^2 \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b (x - \frac{1}{a+b})^2 dx \\ &= \dots \\ &= \frac{(b-a)^2}{12} \end{aligned}$$

Expected Value of Functions of a Random Variable

- Let X be continuous random variable with probability density function $f(x)$.
- If $g(X)$ is a function of the r.v. X, then expected value of $g(X)$ is given by

$$E[g(X)] = \int_{x_{min}}^{x_{max}} g(x)f(x)dx$$

- As an example, let $g(X) = X^2 - 4$ and $f(x)$ as in Example 1, then

$$\begin{aligned} E[g(X)] &= \int_1^5 \left(x^2 - 4\right) f(x)dx \\ &= \int_1^5 \left(x^2 - 4\right) \frac{1}{8}(x-1)dx \\ &= 248/24 \end{aligned}$$

Expected Value of Functions of a Random Variable

- Note that
  - $g(X) = X \quad \Longrightarrow \quad E(g(X)) = E(X) = \mu_x$
  - $g(X) = (X - \mu_x)^2 \quad \Longrightarrow \quad E(g(X)) = E\left[(X - \mu_x)^2\right] = \sigma_x^2$
- When g is just a linear function of X:

$$g(X) = a + bX.$$

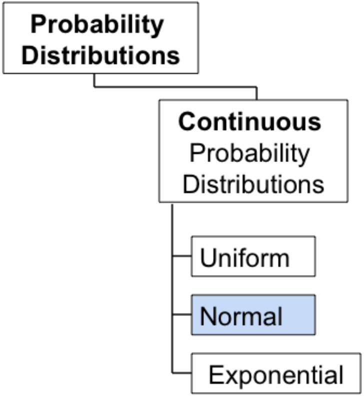
- Remember this is exactly what we called $Y = a + bX$ in Ch 4. (They are just two different ways of expressing the same thing, however writing $g(X) = a + bX$ is mathematically right one. Mathematically $Y(X) = a + bX$ is correct too.)

Result

Let X be cont's r.v. with the expected value μ_x and the variance σ_x^2, then for $Y = a + bX$ we have

- $\mu_y = a + b\mu_x, \quad \left[E(Y) = E(a + bX) = a + bE(X)\right]$
- $\sigma_y^2 = b^2\sigma_x^2, \quad \left[Var(Y) = Var(a + bX) = b^2 Var(X)\right]$ and also $\sigma_y = |b|\sigma_x$

The Normal Distribution



The Normal Distribution

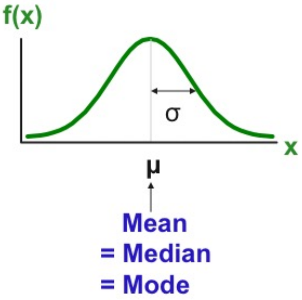
(continued)

- **Bell Shaped**
- **Symmetrical**
- **Mean, Median and Mode are Equal**

Location is determined by the mean, μ

Spread is determined by the standard deviation, σ

The random variable has an infinite theoretical range: + ∞ to - ∞

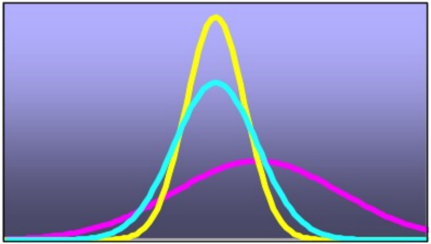


The Normal Distribution

(continued)

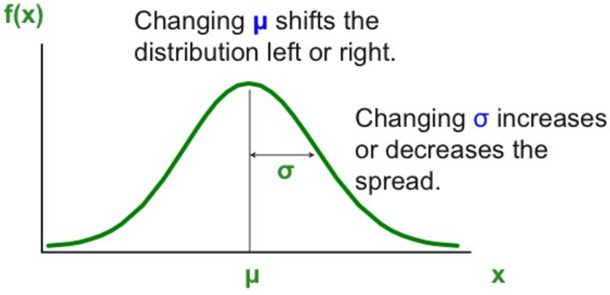
- The normal distribution closely approximates the probability distributions of a wide range of random variables
- Distributions of sample means approach a normal distribution given a “large” sample size
- Computations of probabilities are direct and elegant
- The normal probability distribution has led to good business decisions for a number of applications

Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

The Normal Distribution Shape



Given the mean μ and variance σ^2 we define the normal distribution using the notation $\boxed{X \sim N(\mu, \sigma^2)}$

The Normal Probability Density Function

- The formula for the normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

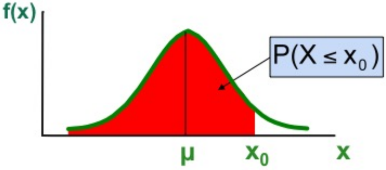
σ^2 = the population variance

x = any value of the continuous variable, $-\infty < x < \infty$

Cumulative Normal Distribution

- For a normal random variable X with mean μ and variance σ^2 , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

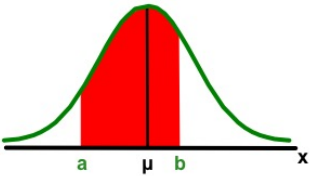
$$\boxed{F(x_0) = P(X \le x_0)}$$



Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

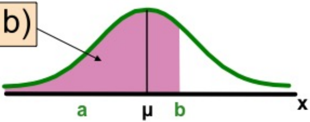
$$\boxed{P(a < X < b) = F(b) - F(a)}$$



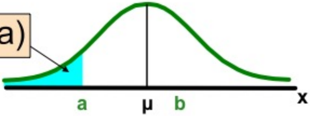
Finding Normal Probabilities

(continued)

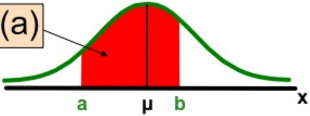
$F(b) = P(X < b)$



$F(a) = P(X < a)$



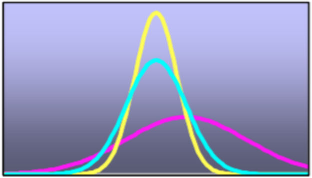
$P(a < X < b) = F(b) - F(a)$



Difficulty of Computing Normal Probabilities

Consider

- $X \backsim N(2, 0.9^2)$
- $Y \backsim N(2, 1.1^2)$
- $T \backsim N(3, 1.5^2)$



- Compute $P(0 < X < 1)$, $P(1 < X < 2)$, $P(2 < X < 3)$?

$$P(0 < X < 1) = \int_0^1 \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} dx = \int_0^1 \frac{e^{\frac{-(x-2)^2}{2(.9)^2}}}{\sqrt{2\pi(.9)^2}} dx$$

- Similarly, $P(1 < X < 2)$, $P(2 < X < 3)$, $P(1 < X < 2)$?
- Compute $P(0 < Y < 1)$,$P(1 < Y < 2)$, $P(2 < Y < 3)$?
- Compute $P(0 < T < 1)$,$P(1 < T < 2)$, $P(2 < T < 3)$?
- Even if we fix the intervals, the integral should be computed for each different r.v. (i.e. different pair (μ, σ^2))
- Too much work! Any practical way to get around?

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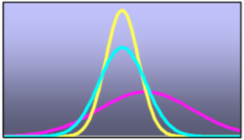
Getting over the Difficulty

Consider now

- $Z \backsim N(0, 1)$
- $X \backsim N(0, 0.5^2)$
- $T \backsim N(3, 1.2^2)$



- Suppose we compute $P(a < Z < b)$,

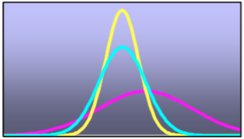
$$P(a < Z < b) = \int_a^b \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} dx = \int_a^b \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} dx$$

for many many different a and b values and store all these probabilities on excel file.

Getting over the Difficulty

Consider now

- $Z \backsim N(0, 1)$, (green)
- $X \backsim N(0, 0.5^2)$, (yellow)
- $T \backsim N(3, 1.2^2)$, (plum)



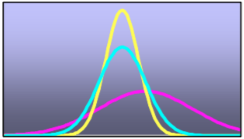
- Do you think we can compute $P(1 < X < 2)$ from these stored probabilities that we computed for Z?
  - Imagine we start deforming the yellow distribution (Z) into the green one (X)
  - Since the area corresponding to $P(1 < X < 2)$ should be maintained, boundaries should move as well
  - Suppose at the moment that the yellow curve overlaps exactly with the green one, 1 becomes c and 2 becomes d, i.e. $c \longleftarrow 1$ and $2 \longrightarrow d$
  - Since green and yellow curves are the same thing now, go to the stored values to bring $P(c < Z < d)$. This is the probability that we wanted to find

$$P(1 < X < 2) = P(c < Z < d)$$

Can we find exact values of c and d?

Consider now

- $Z \backsim N(0,1)$
- $X \backsim N(0,0.5^2)$
- $T \backsim N(3,1.2^2)$



- Let $W = \frac{X}{0.5}$, then

$$E(W) = E[\frac{X}{0.5}] = \frac{1}{0.5}E[X] = 0$$
$$Var(W) = Var[\frac{X}{0.5}] = \frac{1}{0.5^2}Var[X] = 1$$

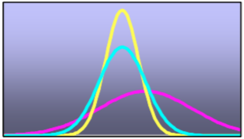
- $W \backsim N(0,1)$, which means $W = Z$, i.e.

$$\frac{X}{0.5} \longrightarrow Z$$

Can we find exact values of c and d?

Consider now

- $Z \backsim N(0,1)$
- $X \backsim N(0,0.5^2)$
- $T \backsim N(3,1.2^2)$



- $W \backsim N(0,1)$, which means $W = Z$, i.e.

$$\frac{X}{0.5} \longrightarrow Z$$

- Then to find c and d s.t.

$$P(1 < X < 2) = P(c < Z < d)$$

- using the previous result we can write

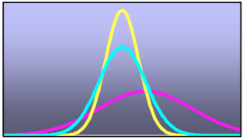
$$c \longleftarrow \frac{1}{0.5} \quad \text{and} \quad \frac{2}{0.5} \longrightarrow d$$

$\implies c = 2$ and $d = 4$.

Can we do the same thing with $T \backsim N(3, 1.2^2)$?

Consider now

- $Z \backsim N(0,1)$
- $X \backsim N(0, 0.5^2)$
- $T \backsim N(3, 1.2^2)$



- Let $W = \frac{T-3}{1.2}$, then

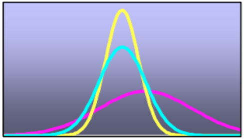
$$
\begin{aligned}
E(W) &= E[\frac{T-3}{1.2}] = \frac{1}{1.2}(E[T-3]) \\
&= \frac{1}{1.2}(E[T]-3) = \frac{1}{1.2}(3-3) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
Var(W) &= Var[\frac{T-3}{1.2}] = \frac{1}{1.2^2} Var[T-3] \\
&= \frac{1}{1.2^2} Var[T] = \frac{1}{(1.2)^2}(1.2)^2 \\
&= 1
\end{aligned}
$$

Can we do the same thing with $T \backsim N(3, 1.2^2)$?

Consider now

- $Z \backsim N(0, 1)$
- $X \backsim N(0, 0.5^2)$
- $T \backsim N(3, 1.2^2)$



- $W \backsim N(0, 1)$, which means $W = Z$. Namely

$$\frac{T-3}{1.2} \longrightarrow Z$$

- As an illustration, let us find k and p s.t.

$$P(1.8 < T < 4.8) = P(k < Z < p)$$

- using the previous result we can write

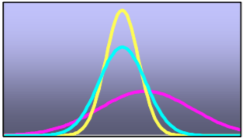
$$k \longleftarrow \frac{1.8-3}{1.2} \quad \text{and} \quad \frac{4.8-3}{1.2} \longrightarrow p$$

$\implies k = -1$ and $p = 1$.

Summary of the last two example and a general result

Consider now

- $Z \sim N(0, 1)$
- $X \sim N(0, 0.5^2)$
- $T \sim N(3, 1.2^2)$



- In the last two example, we have seen that the following transformations convert X and T into Z:

$$\frac{X}{0.5} \longrightarrow Z \quad \text{and} \quad \frac{T-3}{1.2} \longrightarrow Z$$

- Do you see any pattern in these two transformations?
- In both cases, we first subtracted the mean of the rv and then divided by its standard deviation:

$$\frac{X-\mu_x}{\sigma_x} \longrightarrow Z \quad \text{and} \quad \frac{T-\mu_T}{\sigma_T} \longrightarrow Z$$

Z transformation

- Suppose we have a random variable X with mean μ_x and variance σ_x^2, i.e.

$$X \sim N(\mu_x, \sigma_x^2)$$

- Then,

$$Z = \frac{X - \mu_x}{\sigma_x} \sim N(0, 1)$$

  i.e. Z transforms X to a normal distribution with mean 0 and variance 1.
- The distribution $Z \sim N(0, 1)$ is called the standard normal distribution.

Example 2

- Suppose X is distributed normally with mean $\mu_x = 100$ and standard deviation $\sigma_x = 50$.
- Find the Z value for $X = 200$ and $X = 50$.
- For $X = 200$,

$$Z = \frac{X - \mu_x}{\sigma_x} = \frac{200 - 100}{50} = 2.0$$

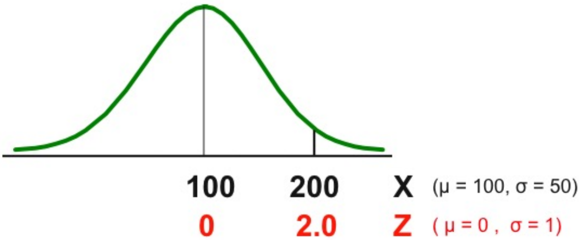
$\rightarrow$ This says that $X = 200$ is two standard deviations (2 increments of 50 units) above the mean of 100.

- For $X = 50$,

$$Z = \frac{X - \mu_x}{\sigma_x} = \frac{50 - 100}{50} = -1.0$$

$\rightarrow$ This says that $X = 50$ is one standard deviation (1 increment of 50 units) below the mean of 100.

Comparing X and Z units



Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

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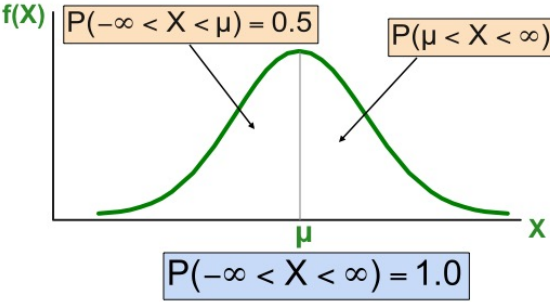
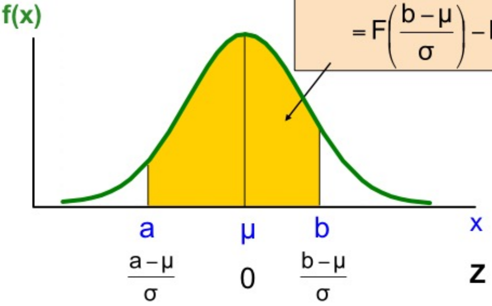
Probability as Area Under the Curve

The total area under the curve is 1.0, and the curve is symmetric, so half is above the mean, half is below



Finding Normal Probabilities

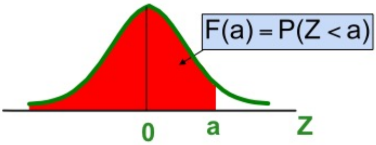
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$
$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$



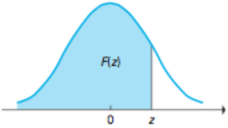
Appendix Table 1

- The Standard Normal Distribution table in the textbook (Appendix Table 1) shows values of the cumulative normal distribution function

- For a given Z-value a , the table shows F(a)
  (the area under the curve from negative infinity to a)



Z-tables 1

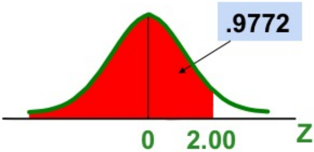


z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817

The Standard Normal Table

- Appendix Table 1 gives the probability F(a) for any value a

Example:
P(Z < 2.00) = .9772



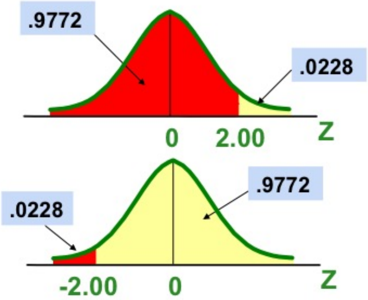
The Standard Normal Table

(continued)

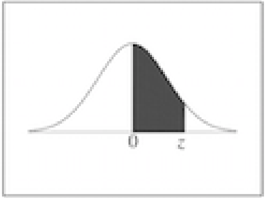
- For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:

Example:

$P(Z < -2.00) = 1 – 0.9772$
$= 0.0228$

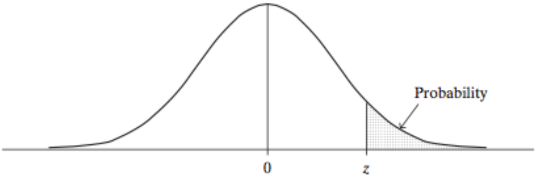


Z-tables 2



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.0000	.0040	.0080	.0120	.0160	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1064	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2517	.2549

Z-tables 3



z	Second Decimal Place of z .00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641
0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451

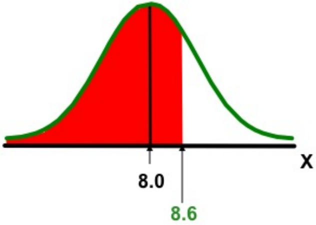
General Procedure for Finding Probabilities

To find $P(a < X < b)$ when X is distributed normally:

- Draw the normal curve for the problem in terms of X
- Translate X-values to Z-values
- Use the Cumulative Normal Table

Finding Normal Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0
- Find P(X < 8.6)

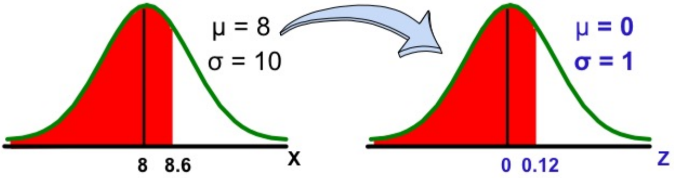


Finding Normal Probabilities

(continued)

- Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)

$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$$



P(X < 8.6)

P(Z < 0.12)

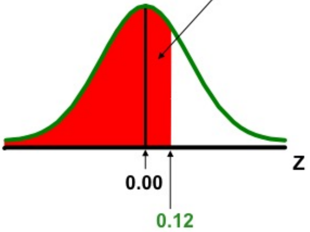
Solution: Finding P(Z < 0.12)

Standardized Normal Probability Table (Portion)

z	F(z)
.10	.5398
.11	.5438
.12	.5478
.13	.5517

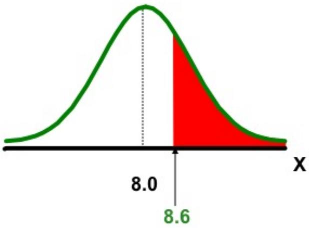
P(X < 8.6)
= P(Z < 0.12)

F(0.12) = 0.5478



Upper Tail Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(X > 8.6)

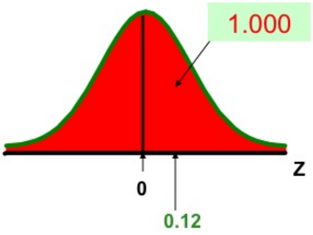


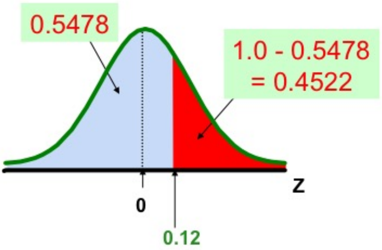
Upper Tail Probabilities

(continued)

- Now Find $P(X > 8.6)$…

$$P(X > 8.6) = P(Z > 0.12) = 1.0 - P(Z \leq 0.12)$$
$$= 1.0 - 0.5478 = \boxed{0.4522}$$





Finding the X value for a Known Probability

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

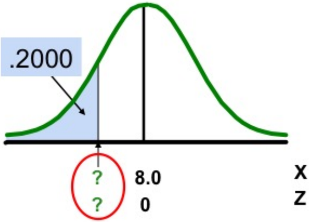
$$\boxed{X = \mu + Z\sigma}$$

Finding the X value for a Known Probability

(continued)

Example:

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now find the X value so that only 20% of all values are below this X



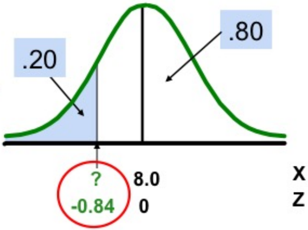
Find the Z value for 20% in the Lower Tail

1. Find the Z value for the known probability

Standardized Normal Probability Table (Portion)

z	F(z)
.82	.7939
.83	.7967
.84	.7995
.85	.8023

- 20% area in the lower tail is consistent with a Z value of -0.84



Finding the X value

2. Convert to X units using the formula:

$$X = \mu + Z\sigma$$
$$= 8.0 + (-0.84)5.0$$
$$= 3.80$$

So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80

Normal Distribution Approximation for Binomial Distribution

- Recall the binomial distribution:
  - n independent trials
  - probability of success on any given trial = P
- Random variable X:
  - X_i =1 if the i^{th} trial is "success"
  - X_i =0 if the i^{th} trial is "failure"

$$\boxed{E[X] = \mu = nP}$$

$$\boxed{Var(X) = \sigma^2 = nP(1-P)}$$

Normal Distribution Approximation for Binomial Distribution

(continued)

- The shape of the binomial distribution is approximately normal if n is large
- The normal is a good approximation to the binomial when $nP(1-P) > 5$
- Standardize to Z from a binomial distribution:

$$\boxed{Z = \frac{X - E[X]}{\sqrt{Var(X)}} = \frac{X - np}{\sqrt{nP(1-P)}}}$$

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Normal Distribution Approximation for Binomial Distribution

(continued)

- Let X be the number of successes from n independent trials, each with probability of success P.
- If $nP(1 - P) > 5$,

$$P(a < X < b) = P\left(\frac{a - nP}{\sqrt{nP(1-P)}} \le Z \le \frac{b - nP}{\sqrt{nP(1-P)}}\right)$$

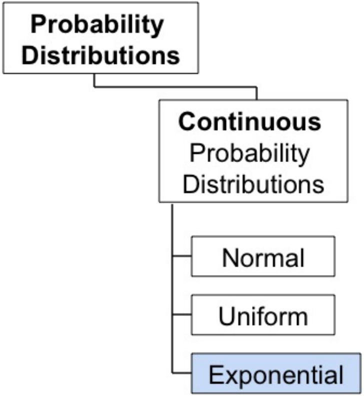
Binomial Approximation Example

- 40% of all voters support ballot proposition A. What is the probability that between 76 and 80 voters indicate support in a sample of n = 200 ?
  - $E[X] = \mu = nP = 200(0.40) = 80$
  - $Var(X) = \sigma^2 = nP(1-P) = 200(0.40)(1-0.40) = 48$

    (note: $nP(1-P) = 48 > 5$)

$$
\begin{aligned}
P(76 < X < 80) &= P\left(\frac{76-80}{\sqrt{200(0.4)(1-0.4)}} \le Z \le \frac{80-80}{\sqrt{200(0.4)(1-0.4)}}\right) \\
&= P(-0.58 < Z < 0) \\
&= F(0) - F(-0.58) \\
&= 0.5000 - 0.2810 = 0.2190
\end{aligned}
$$

The Exponential Distribution

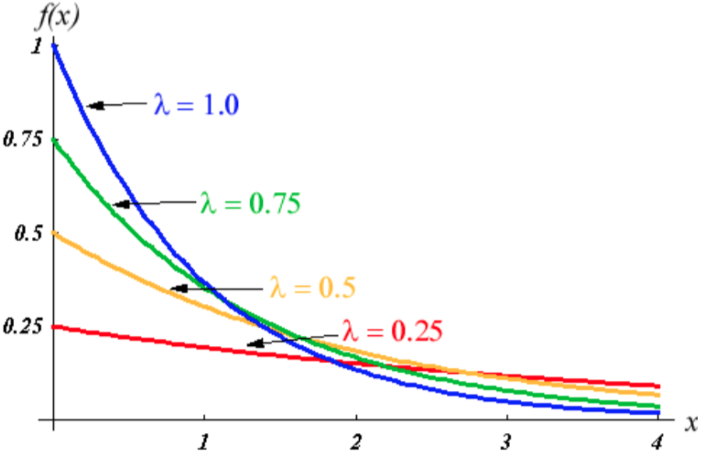


The Exponential Distribution

- Used to model the length of time between two occurrences of an event (the time arrivals)
- Examples
  - Time between trucks arriving at an unloading dock
  - Time between transactions at an ATM machine
  - Time between phone calls to the main operator
- The exponential random variable T has a probability density function

$$f(t) = \lambda e^{-\lambda t} \quad \text{for } t > 0$$

  where,
  - λ is the mean number of occurrences per unit time
  - t is the number of time units until the next occurrence
  - $e = 2.718$
- Defined by a single parameter, its mean λ


f(x)
1
0.75
0.5
0.25
λ = 1.0
λ = 0.75
λ = 0.5
λ = 0.25
1
2
3
4
x

The Exponential Distribution cont'd

- The cumulative distribution function (the probability that an arrival time is less than some specified time t) is

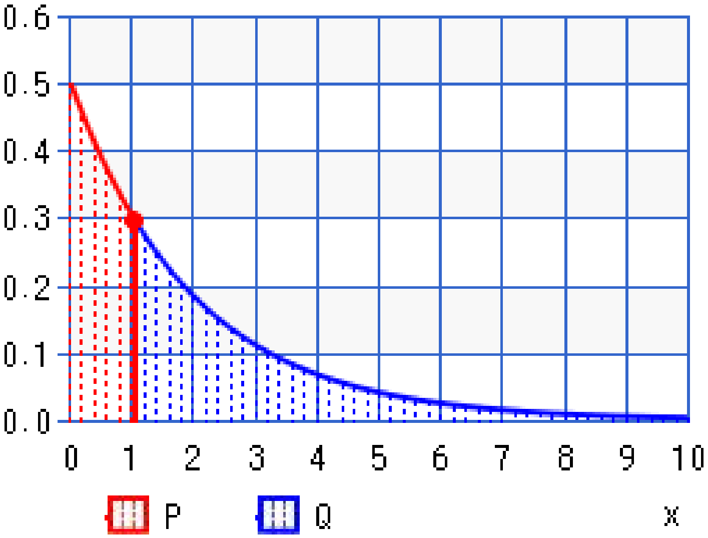
$$F(t) = 1 - e^{-\lambda t} \quad \text{for } t > 0$$

  where,

  - λ is the mean number of occurrences per unit time
  - $e = 2.718$

- In another words, with probability $F(t)$, it will take at most t time units to observe the next occurrence, i.e.

$$F(t) = P(T < t)$$


0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
1
2
3
4
5
6
7
8
9
10
P
Q
x

Example 3

Suppose customers arrive at the service counter at the rate of 15 per hour.

- What is the probability that the arrival time between consecutive customers is less than three minutes?
  - The mean number of arrivals per hour is 15, so $\lambda = 15$
  - Three minutes is .05 hours
  - $P(T < .05) = 1 - e^{-\lambda x} = 1 - e^{-(15)(0.05)} = 0.5276 \rightarrow$ There is a 52.76% probability that the arrival time between successive customers is less than three minutes
- Suppose that one customer has just left. What is the probability that in order to see the next customer we will have to wait two to four minutes?

$$\begin{aligned} P(2 < T < 4) &= P(T < 4) - P(T < 2) \\ &= -e^{-(15)(4/60)} + e^{-(15)(2/60)} \\ &= 0.239 \end{aligned}$$

Definitions

- A joint probability density function of two continuous r.v.'s denoted by
$$f(x, y)$$
- However, just like $f(x)$ is not the probability of $X = x$, $f(x, y)$ is not the probability of $(X = x, Y = y)$.
- Recall that in single variable case probabilities were defined only for intervals, not for individual values. In the same way, joint probability density function $f(x, y)$ is defined only for areas.
- We are not going to study the continuous analogs of marginal and conditional probability distributions.

Covariance and Correlation

- Let X and Y be two continuous r.v.'s with means μ_x and μ_y, respectively.
- The expected value of $(X-\mu_x)(Y-\mu_y)$ is called covariance between X and Y:

$$
\begin{aligned}
Cov(X,Y) &= E\left[(X-\mu_x)(Y-\mu_y)\right] \\
&= \int_x \int_y (x-\mu_x)(y-\mu_y) f(x,y)\,dxdy
\end{aligned}
$$

- If two random variables are statistically independent, the covariance between them is 0. However, the converse is not necessarily true unless random variables are normally distributed.
- Correlation coefficient measures both the direction and strength of the linear relationship between two r.v. variables

$$\rho = Corr(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

Linear Functions of Two Random Variables

- For two continuous r.v.'s X and Y the following linear combination:

$$W = aX \pm bY$$

defines a new r.v. just like in the discrete case.

- The Expected Value for the linear combination of two r.v.'s

$$E[W] = aE(X) \pm bE(Y) \quad \text{or}$$
$$\mu_w = a\mu_x \pm b\mu_y$$

- The Variance for the linear combination of two r.v.'s

$$Var[W] = a^2 Var(X) + b^2 Var(Y) \pm 2abCov(X, Y) \quad \text{or}$$
$$\sigma_w^2 = a^2\sigma_x^2 + b^2\sigma_y^2 \pm 2abCov(X, Y)$$

Linear Functions of Two Random Variables

- Consider again a linear combination X and Y :

$$W = aX \pm bY$$

- In general, even if we manage to compute $E(W)$ and $Var(W)$, we cannot say much about the shape of the distribution.
- However, when X and Y are normal r.v.'s , then W is normally distributed as well

$$W \backsim N(\mu_w, \sigma_w^2)$$

- This is a very important property because it basically allows us to compute all kinds of probabilities that we might be interested in regarding W.

Example 4

- Two tasks, X and Y, must be performed by the same worker
  - The worker completes X in 20 minutes on average, with a standard deviation of 5.
  - The worker completes Y in 30 minutes on average, with a standard deviation of 8.
  - Suppose X and Y are normally distributed and independent.
- What is the probability that the worker will complete both tasks in less than 45 minutes?
  - Define $W = X + Y$
  - Then

$$\begin{aligned} \mu_w &= \mu_x + \mu_y \\ &= 20 + 30 \\ &= 50 \end{aligned} \qquad \begin{aligned} \sigma_w^2 &= \sigma_x^2 + \sigma_y^2 + 2Cov(X, Y) \\ &= 5^2 + 8^2 + 2.0 \\ &= 89 \end{aligned}$$

  - Therefore, $W \sim N(50, 89)$, and finally

$$\begin{aligned} P(W < 45) &= P(Z < \frac{45 - 50}{\sqrt{89}}) \\ &= P(Z < -0.535) \approx 0.3 \end{aligned}$$

Example 5

- Consider two stocks, A and B.
  - The price of stock A is normally distributed with mean 12 and standard deviation of 4.
  - The price of stock B is normally distributed with mean 20 and standard deviation of 16.
  - Stock prices are positively correlated with correlation $\rho_{AB} = 0.5$.
- Suppose you own 10 shares of A and 30 shares of B.
- What is the probability that your portfolio value will be less than \$ 500?

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Example 5 cont'd

What is the probability that your portfolio value will be less than \$ 500?

- Define $W = 10A + 30B$
- Then,

$$\begin{aligned}
\mu_w &= 10\mu_A + 30\mu_B \\
&= (10)(12) + (30)(20) \\
&= 720 \\
\sigma_w^2 &= 10^2\sigma_A^2 + 30^2\sigma_B^2 + 2(10)(30)Cov(X, Y) \\
&= 10^2(4^2) + 30^2(16^2) + 2(10)(30)(0.5)(4)(16) \\
&= 251,200
\end{aligned}$$

- Therefore, $W \sim N(720, 251\,200)$, and finally

$$\begin{aligned}
P(W < 500) &= P(Z < \frac{500 - 720}{\sqrt{251,200}}) \\
&= P(Z < -0.44) \approx 0.33
\end{aligned}$$

$\rightarrow$ So the probability is 0.33 that your portfolio value will be less than \$500.